1. Solve the following system by Gaussian Elimination (without row interchanges) and back substitution:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 8 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 1 \\ 3x_1 & - & 7x_2 & + & 4x_3 & = & 10 \end{array}$$

solution:

Augmented Matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

Eliminate in first column:

$$\begin{array}{rc} & \\ R_2 + R_1 & \longrightarrow \\ R_3 - 3R_1 & \longrightarrow \end{array} \quad \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix}$$

Eliminate in second column:

$$\begin{array}{rc} & \\ & \\ R_3 - 10R_2 & \longrightarrow \end{array} \quad \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{bmatrix}$$

Elimination completion. Matrix is in echelon (upper triangular) form. Back substitute:

$$x_3 = \frac{-104}{-52} = 2$$

$$x_2 = \frac{9 - 5x_3}{-1} = \frac{9 - (5)(2)}{-1} = \frac{-1}{-1} = 1$$

$$x_1 = \frac{8 - x_2 - 2x_3}{1} = \frac{8 - 1 - (2)(2)}{1} = \frac{3}{1} = 3$$

2. Solve the following system by Gaussian Elimination (with row interchanges to avoid zero pivots only) and back substitution:

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & 2x_3 & = & 1 \\ 6x_1 & - & 2x_2 & + & 5x_3 & = & -1 \\ -3x_1 & + & 2x_2 & - & x_3 & = & -5 \end{array}$$

solution:

Augmented Matrix:

$$\begin{bmatrix} 3 & -1 & 2 & 1 \\ 6 & -2 & 5 & -1 \\ -3 & 2 & -1 & -5 \end{bmatrix}$$

Eliminate in first column:

$$\begin{array}{rl} & \\ R_2 - 2R_1 & \longrightarrow \\ R_3 + R_1 & \longrightarrow \end{array} \qquad \begin{bmatrix} 3 & -1 & 2 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & 1 & -4 \end{bmatrix}$$

Zero on diagonal in second column mandates row interchange.

$$\begin{array}{rl} & \\ R_2 \leftrightarrow R_3 & \longrightarrow \\ R_2 \leftrightarrow R_3 & \longrightarrow \end{array} \qquad \begin{bmatrix} 3 & -1 & 2 & 1 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Elimination completion. Matrix is in echelon (upper triangular) form. Back substitute:

$$x_3 = \frac{-3}{1} = -3$$

$$x_2 = \frac{-4 - x_3}{1} = \frac{-4 - (-3)}{1} = -1$$

$$x_1 = \frac{1 + x_2 - 2x_3}{3} = \frac{1 + (-1) - (2)(-3)}{3} = \frac{6}{3} = 2$$

3. Consider the following matrix (unaugmented):

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 3 & 1 \\ 2 & 1 & -2 & 5 & 3 \\ -2 & 1 & 6 & -6 & 1 \end{bmatrix}$$

a. Using elementary row operations (with row interchanges only when necessary to remove zeros in pivot positions), reduce the appropriate matrix to echelon form, and determine for what right-hand sides ($\mathbf{b}$) the system:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

will be solvable.

solution:

For any right-hand side, the augmented matrix will have the form.

$$[\,\mathbf{A} \vdots \mathbf{b}\,] = \left[\begin{array}{ccccccc} 1 & 0 & -2 & 3 & 1 & \vdots & b_1 \\ 2 & 1 & -2 & 5 & 3 & \vdots & b_2 \\ -2 & 1 & 6 & -6 & 1 & \vdots & b_3 \end{array}\right]$$

Proceed by eliminating in the first column:

$$\begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + 2R_1 \end{array} \quad \longrightarrow \quad \left[\begin{array}{ccccccc} 1 & 0 & -2 & 3 & 1 & \vdots & b_1 \\ 0 & 1 & 2 & -1 & 1 & \vdots & b_2 - 2b_1 \\ 0 & 1 & 2 & 0 & 3 & \vdots & b_3 + 2b_1 \end{array}\right]$$

There is a pivot in the second column, so eliminate in that column:

$$\begin{array}{c} \\ \\ R_3 - R_2 \end{array} \quad \longrightarrow \quad \left[\begin{array}{ccccccc} 1 & 0 & -2 & 3 & 1 & \vdots & b_1 \\ 0 & 1 & 2 & -1 & 1 & \vdots & b_2 - 2b_1 \\ 0 & 0 & 0 & 1 & 2 & \vdots & b_3 - b_2 + 4b_1 \end{array}\right]$$

Elimination completion. Matrix is in echelon (upper triangular) form. Observe there is a pivot element on each row. Therefore, the system will always be solvable.

b. What is the general solution for the homogeneous problem associated with this system?

solution:

The homogeneous problem is $\mathbf{A}\,\mathbf{x} = \mathbf{0}$. For this problem the augmented matrix is

$$[\,\mathbf{A} \vdots \mathbf{0}\,] = \left[\begin{array}{rrrrrcr} 1 & 0 & -2 & 3 & 1 & \vdots & 0 \\ 2 & 1 & -2 & 5 & 3 & \vdots & 0 \\ -2 & 1 & 6 & -6 & 2 & \vdots & 0 \end{array}\right]$$

and therefore, using the results obtained above for $b_i \equiv 0$, will have the following echelon form:

$$\left[\begin{array}{rrrrrcr} 1 & 0 & -2 & 3 & 1 & \vdots & 0 \\ 0 & 1 & 2 & -1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 2 & \vdots & 0 \end{array}\right]$$

We can now backsolve. The fifth column is free, so we can set

$$x_5 = \alpha \quad \text{(arbitrary)}$$

then

$$x_4 = 0 - 2x_5 = -2\alpha$$

But third column is also free, so

$$x_3 = \beta \quad \text{(arbitrary)}$$

and then

$$x_2 = -2x_3 + x_4 - x_5 = -(2)(\beta) + (-2\alpha) - (\alpha) = -3\alpha - 2\beta$$

and finally

$$x_1 = 2x_3 - 3x_4 - x_5 = (2)(\beta) - 3(-2\alpha) - (\alpha) = 5\alpha + 2\beta$$

or equivalently:

$$\mathbf{x} = \left[\begin{array}{c} 5\alpha + 2\beta \\ -3\alpha - 2\beta \\ \beta \\ -2\alpha \\ \alpha \end{array}\right] = \alpha \left[\begin{array}{r} 5 \\ -3 \\ 0 \\ -2 \\ 1 \end{array}\right] + \beta \left[\begin{array}{r} 2 \\ -2 \\ 1 \\ 0 \\ 0 \end{array}\right]$$

c. What is the rank of $\mathbf{A}$?

solution:

As shown above, when the augmented matrix $[\ \mathbf{A} \ \vdots \ \mathbf{b}\ ]$ is reduced to echelon form, the third and fifth columns are free, while the first, second and fourth are pivot (basic) columns. Since $\mathbf{A}$ has only three pivot columns, then

$$\text{rank}(\mathbf{A}) = 3$$

d. What is the Row Reduced Echelon Form of $\mathbf{A}$?

solution:

In solveing the homogeneous problem, we saw that the augmented matrix associated with $\mathbf{A}$ could be reduced by Gaussian Elimination to the echelon form:

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 1 & \vdots & 0 \\ 0 & 1 & 2 & -1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 2 & \vdots & 0 \end{bmatrix}$$

We can no proceed to eliminate the elements in the columns above the each of the pivot positions. (We could also have done this during the elimination!) This produces:

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 1 & \vdots & 0 \\ 0 & 1 & 2 & -1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 2 & \vdots & 0 \end{bmatrix} \quad \begin{array}{l} R_1 - 3R_3 \longrightarrow \\ R_2 + R_3 \quad \longrightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & -2 & 0 & -5 & \vdots & 0 \\ 0 & 1 & 2 & 0 & 3 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 2 & \vdots & 0 \end{bmatrix}$$

Since there are already zeros everywhere above the pivot in the second column, we're done. The row reduced echelon form of $\mathbf{A}$ is:

$$\begin{bmatrix} 1 & 0 & -2 & 0 & -5 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

e. Express each non-pivot (free) column in the row reduced echelon form as a linear combination of the pivot columns.

solution:

The third column of the row reduced echelon matrix can be written:

$$\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = -2 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{1^{st} \text{ col}} +2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{2^{nd} \text{ col}}$$

while the fifth column can be written:

$$\begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix} = -5 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{1^{st} \text{ col}} +3 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{2^{nd} \text{ col}} +2 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{4^{th} \text{ col}}$$

which shows how, in this specific case, the free columns of the row reduced echelon form can be construced as linear combinations of the pivot columns.

4. Consider the following matrix (unaugmented):

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 4 & 4 \\ -3 & -6 & 3 & -5 & -8 \\ 2 & 4 & -2 & 5 & 2 \end{bmatrix}$$

a. Using elementary row operations (with row interchanges only when necessary to remove zeros in pivot positions), reduce the appropriate matrix to echelon form, and determine for what right-hand sides ($\mathbf{b}$) the system:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

will be solvable.

solution:

For any right-hand side, the augmented matrix will have the form.

$$[\,\mathbf{A} \vdots \mathbf{b}\,] = \begin{bmatrix} 2 & 4 & -2 & 4 & 4 & \vdots & b_1 \\ -3 & -6 & 3 & -5 & -8 & \vdots & b_2 \\ 2 & 4 & -2 & 5 & 2 & \vdots & b_3 \end{bmatrix}$$

Proceed by eliminating in the first column:

$$\begin{array}{c} \\ R_2 + \frac{3}{2}R_1 \\ R_3 - R_1 \end{array} \quad \longrightarrow \quad \begin{bmatrix} 2 & 4 & -2 & 4 & 4 & \vdots & b_1 \\ 0 & 0 & 0 & 1 & -2 & \vdots & b_2 + \frac{3}{2}b_1 \\ 0 & 0 & 0 & 1 & -2 & \vdots & b_3 - b_1 \end{bmatrix}$$

The second and third columns are now obviously free, so proceed to eliminate in the fourth column:

$$\begin{array}{c} \\ \\ R_3 - R_2 \end{array} \quad \longrightarrow \quad \begin{bmatrix} 2 & 4 & -2 & 4 & 4 & \vdots & b_1 \\ 0 & 0 & 0 & 1 & -2 & \vdots & b_2 + \frac{3}{2}b_1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & b_3 - b_2 - \frac{5}{2}b_1 \end{bmatrix}$$

Elimination completion. Matrix is in echelon (upper triangular) form.

solution:

a. (cont) Observe there is no pivot element in the third row. Therefore, the system will be solvable, i.e. a solution will *exist*, if and only if the term in corresponding position of the augmented column is identically zero, i.e. if and only if:

$$b_3 - b_2 - \frac{5}{2} b_1 = 0$$

b. What is the general solution for the homogeneous problem associated with this system?

solution:

The homogeneous problem is $\mathbf{A}\,\mathbf{x} = \mathbf{0}$. For this problem the augmented matrix is

$$[\,\mathbf{A} \vdots \mathbf{0}\,] = \begin{bmatrix} 2 & 4 & -2 & 4 & 4 & \vdots & 0 \\ -3 & -6 & 3 & -5 & -8 & \vdots & 0 \\ 2 & 4 & -2 & 5 & 2 & \vdots & 0 \end{bmatrix}$$

and therefore, using the results obtained above for $b_i \equiv 0$, will have the following echelon form:

$$\begin{bmatrix} 2 & 4 & -2 & 4 & 4 & \vdots & 0 \\ 0 & 0 & 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

We can now backsolve. The fifth column is free, so we can set

$$x_5 = \alpha \quad \text{(arbitrary)}$$

then

$$x_4 = 0 + 2x_5 = 2\alpha$$

But the second and third columns are also free, so

$$x_3 = \beta \quad \text{(arbitrary)}$$
$$x_2 = \gamma \quad \text{(arbitrary)}$$

solution:

b. (cont) and finally

$$\begin{aligned} x_1 = \frac{-4x_2 + 2x_3 - 4x_4 - 4x_5}{2} &= -2x_2 + x_3 - 2x_4 - 2x_5 \\ &= (-2)(\gamma) + (\beta) - (2)(2\alpha) - (2)(\alpha) = -6\alpha + \beta - 2\gamma \end{aligned}$$

or equivalently:

$$\mathbf{x} = \begin{bmatrix} -6\alpha + \beta - 2\gamma \\ \gamma \\ \beta \\ 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -6 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

c. What is the rank of **A**?

solution:

As shown above, when the augmented matrix $[\ \mathbf{A} \ \vdots \ \mathbf{b}\ ]$ is reduced to echelon form, the second,third and fifth columns are free, while the first and fourth are pivot (basic) columns. Since **A** has only two pivot columns, then

$$\text{rank}(\mathbf{A}) = 2$$

5. Find the general solution to the following linear system:

$$\begin{array}{rcrcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & 3 \\ -3x_1 & + & 3x_2 & - & 3x_3 & - & 3x_4 & - & 12x_5 & = & 0 \\ 2x_1 & - & 2x_2 & + & 3x_3 & + & 2x_4 & + & 6x_5 & = & 3 \end{array}$$

solution:

The augmented matrix here will have the form:

$$\left[\begin{array}{rrrrrcr} 1 & -1 & 2 & 1 & 2 & \vdots & 3 \\ -3 & 3 & -3 & -3 & -12 & \vdots & 0 \\ 2 & -2 & 3 & 2 & 6 & \vdots & 3 \end{array}\right]$$

Proceed by eliminating in the first column:

$$\begin{array}{l} \\ R_2 + 3R_1 \\ R_3 - 2R_1 \end{array} \quad \longrightarrow \quad \left[\begin{array}{rrrrrcr} 1 & -1 & 2 & 1 & 2 & \vdots & 3 \\ 0 & 0 & 3 & 0 & -6 & \vdots & 9 \\ 0 & 0 & -1 & 0 & 2 & \vdots & -3 \end{array}\right]$$

The second column is now obviously free, so proceed to eliminate in the third column:

$$\begin{array}{l} \\ \\ R_3 - \frac{1}{3}R_2 \end{array} \quad \longrightarrow \quad \left[\begin{array}{rrrrrcr} 1 & -1 & 2 & 1 & 2 & \vdots & 3 \\ 0 & 0 & 3 & 0 & -6 & \vdots & 9 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{array}\right]$$

Elimination completion. Matrix is in echelon (upper triangular) form. Note that even though the third contains **no** pivot, the system is still solvable since the corresponding element in the augmented column is also a zero. (Generally that will not be the case when a row contains no pivot!) So we can back substitute.

We can now backsolve. The fourth and fifth columns are free, so we can set

$$\begin{aligned} x_5 &= \alpha \quad \text{(arbitrary)} \\ x_4 &= \beta \quad \text{(arbitrary)} \end{aligned}$$

and then

$$x_3 = \frac{9 + 6x_5}{3} = 3 + 2x_5 = 3 + 2\alpha$$

solution:

(cont) But the second column is also free, and so

$$\begin{aligned} x_2 &= \gamma \quad \text{(arbitrary)} \\ x_1 &= 3 + x_2 - 2x_3 - x_4 - 2x_5 \\ &= 3 + (\gamma) - (2)(3+2\alpha) - (\beta) - (2)(\alpha) = -3 - 6\alpha - \beta + \gamma \end{aligned}$$

or equivalently:

$$\mathbf{x} = \begin{bmatrix} -3-6\alpha-\beta+\gamma \\ \gamma \\ 3+2\alpha \\ \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -6 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

What is the rank of the matrix associated with this system?

solution:

As shown above, Gaussian Elimination produces

$$\left[\begin{array}{rrrrr:r} 1 & -1 & 2 & 1 & 2 & 3 \\ -3 & 3 & -3 & -3 & -12 & 0 \\ 2 & -2 & 3 & 2 & 6 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{rrrrr:r} 1 & -1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 3 & 0 & -6 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

By definition, the second, fourth and fifth columns are free, while the first and third contain pivots. Therefore, $\mathbf{A}$ contains two pivot (basic) columns, and hence

$$\text{rank}(\mathbf{A}) = 2$$

6. Consider the block matrices:

$$\mathbf{A} = \begin{bmatrix} 4 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -2 & \vdots & 3 & -4 \\ 1 & & 2 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & -2 & \vdots & -5 \\ \cdots & \cdots & \cdots & \cdots \\ 5 & 4 & \vdots & -4 \\ -6 & -7 & & -4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & \vdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & 2 & 0 \\ -2 & & -3 & 2 \end{bmatrix}$$

Compute, if possible the following as both block matrix operations and "ordinary" operations and confirm that the same results occur. If a computation is not possible, explain why.

a. $\mathbf{A}\,\mathbf{B}$

solution:

By definition:

$$\mathbf{A}\,\mathbf{B} = \begin{bmatrix} \mathbf{A}_{11} & \vdots & \mathbf{A}_{12} \\ \cdots & \cdots & \cdots \\ \mathbf{A}_{21} & \vdots & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \vdots & \mathbf{B}_{12} \\ \cdots & \cdots & \cdots \\ \mathbf{B}_{21} & \vdots & \mathbf{B}_{22} \end{bmatrix}$$

This product can be computed. The element $(\mathbf{A}\,\mathbf{B})_{11}$ would be:

$$\mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} = [4]\,[\,-1 \quad -2\,] + [\,0 \quad 0\,] \begin{bmatrix} 5 & 4 \\ -6 & -7 \end{bmatrix} = [\,-4 \quad -8\,]$$

Similarly

$$(\mathbf{A}\,\mathbf{B})_{12} = \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} = [4]\,[-5] + [\,0 \quad 0\,] \begin{bmatrix} -4 \\ -4 \end{bmatrix} = -20$$

$$(\mathbf{A}\,\mathbf{B})_{21} = \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} [\,-1 \quad -2\,] + \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ -6 & -7 \end{bmatrix}$$
$$= \begin{bmatrix} 41 & 44 \\ -9 & -15 \end{bmatrix}$$

$$(\mathbf{A}\,\mathbf{B})_{22} = \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} [-5] + \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 14 \\ -25 \end{bmatrix}$$

solution:

a. (cont.) and therefore:

$$\mathbf{A}\,\mathbf{B} = \left[\begin{array}{cc:c} -4 & -8 & -20 \\ \hdashline 41 & 44 & 14 \\ -9 & -15 & -25 \end{array}\right]$$

This is identical to the MATLAB result provided by ordinary multiplication.

b. $\mathbf{B}\,\mathbf{A}$

solution:

By definition:

$$\mathbf{B}\,\mathbf{A} = \left[\begin{array}{c:c} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \hdashline \mathbf{B}_{21} & \mathbf{B}_{22} \end{array}\right] \left[\begin{array}{c:c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hdashline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right]$$

This product cannot be computed. The element $(\mathbf{B}\,\mathbf{A})_{11}$ would be:

$$\mathbf{B}_{11}\mathbf{A}_{11} + \mathbf{B}_{12}\mathbf{A}_{21} = [\,-1 \quad -2\,]\,(4) + [-5]\begin{bmatrix} -2 \\ 1 \end{bmatrix} = [\,-4 \quad -8\,] + \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

and these last two quantities are incompatible for addition.

c. $\mathbf{A}\,\mathbf{C}$

solution:

By definition: $$\mathbf{A}\,\mathbf{C} = \left[\begin{array}{c:c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hdashline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right] \left[\begin{array}{c:c} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \hdashline \mathbf{C}_{21} & \mathbf{C}_{22} \end{array}\right]$$

solution:

c. (cont) This product can be computed. The element $(\mathbf{A\,C})_{11}$ would be:

$$\mathbf{A}_{11}\mathbf{C}_{11} + \mathbf{A}_{12}\mathbf{C}_{21} = [4]\,[2] + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = 8$$

Similarly

$$(\mathbf{A\,C})_{12} = \mathbf{A}_{11}\mathbf{C}_{12} + \mathbf{A}_{12}\mathbf{C}_{22} = [4] \begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 4 \end{bmatrix}$$

$$(\mathbf{A\,C})_{21} = \mathbf{A}_{21}\mathbf{C}_{11} + \mathbf{A}_{22}\mathbf{C}_{21} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} [2] + \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

$$(\mathbf{A\,C})_{22} = \mathbf{A}_{21}\mathbf{C}_{12} + \mathbf{A}_{22}\mathbf{C}_{22} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 16 & -10 \\ -4 & 7 \end{bmatrix}$$

and therefore:

$$\mathbf{A\,B} = \begin{bmatrix} 8 & \vdots & 4 & 4 \\ \cdots & \cdots & \cdots & \cdots \\ 4 & \vdots & 16 & -10 \\ -4 & & -4 & 7 \end{bmatrix}$$

This is identical to the MATLAB result provided by ordinary multiplication.

d. $\mathbf{A} + \mathbf{B}$

solution:

By definition: $$\mathbf{A} + \mathbf{B} = \begin{bmatrix} \mathbf{A}_{11} & \vdots & \mathbf{A}_{12} \\ \cdots & \cdots & \cdots \\ \mathbf{A}_{21} & \vdots & \mathbf{A}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{11} & \vdots & \mathbf{B}_{12} \\ \cdots & \cdots & \cdots \\ \mathbf{B}_{21} & \vdots & \mathbf{B}_{22} \end{bmatrix}$$

solution:

d. (cont). This sum cannot be computed. The element $(\mathbf{A}+\mathbf{B})_{11}$ would be:

$$\mathbf{A}_{11}+\mathbf{B}_{11} = [4] + [\,-1 \quad -2\,]$$

and these are incompatible for addition.

e. $\mathbf{A}+\mathbf{C}$

solution:

By definition: $$\mathbf{A}+\mathbf{C} = \begin{bmatrix} \mathbf{A}_{11} & \vdots & \mathbf{A}_{12} \\ \cdots & \cdots & \cdots \\ \mathbf{A}_{21} & \vdots & \mathbf{A}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{11} & \vdots & \mathbf{C}_{12} \\ \cdots & \cdots & \cdots \\ \mathbf{C}_{21} & \vdots & \mathbf{C}_{22} \end{bmatrix}$$

This sum can be computed. The element $(\mathbf{A}+\mathbf{C})_{11}$ would be:

$$\mathbf{A}_{11}+\mathbf{C}_{11} = [4] + [2] = 6$$

Similarly

$$(\mathbf{A}+\mathbf{C})_{12} = \mathbf{A}_{12}+\mathbf{C}_{12} = [\;0 \quad 0\,] + [\;1 \quad 1\,] = [\;1 \quad 1\,]$$

$$(\mathbf{A}+\mathbf{C})_{21} = \mathbf{A}_{21}+\mathbf{C}_{21} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$(\mathbf{A}+\mathbf{C})_{22} = \mathbf{A}_{22}+\mathbf{C}_{22} = \begin{bmatrix} 3 & -4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -1 & 5 \end{bmatrix}$$

and therefore:

$$\mathbf{A}+\mathbf{C} = \begin{bmatrix} 6 & \vdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -2 & \vdots & 5 & -4 \\ -1 & \vdots & -1 & 5 \end{bmatrix}$$

This is identical to the MATLAB result provided by ordinary addition.

7. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix}$$

a. Find a sequence of elementary matrices which, when successively multiplying $\mathbf{A}$ on the left, reduce $\mathbf{A}$ to row-reduced echelon form (or, equivalently, which produce the same result as Gauss-Jordan elimination.)

solution:

Step 1: Starting with the original matrix,i.e.: $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix}$

Step 2: Use Gauss-Jordan elimination to produce Reduced Row-Echelon Form (RREF):

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix} \; R_3 + 2R_1 \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & -4 \end{bmatrix}$$

But this should be the same as multiplying the original augmented matrix on the left by the elementary matrix produced by applying the identical row operation to the identity, i.e.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; R_3 + 2R_1 \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \equiv \mathbf{E}^{(1)}$$

and

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(1)}} \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix}}_{\mathbf{A}} = \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & -4 \end{bmatrix}}_{\mathbf{E}^{(1)}\mathbf{A}}$$

solution:

a. (cont) The next elementary row operation would be to replace the third row by $R_3 - \frac{1}{2}R_2$, which should be equivalent to multiplying on the left by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_3 - (1/2)R_1 \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \equiv \mathbf{E}^{(2)}$$

and therefore the elimination should produce:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}}_{\mathbf{E}^{(2)}} \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 1 & -4 \end{bmatrix}}_{\mathbf{E}^{(1)}\mathbf{A}} = \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -6 \end{bmatrix}}_{\mathbf{E}^{(2)}\mathbf{E}^{(1)}\mathbf{A}}$$

With the matrix in echelon form, back substitute to row reduced echelon form. First replace the third row by $-(1/6)R_3$, or multiply on the left by:

$$\mathbf{E}^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{6} \end{bmatrix}$$

i.e.:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{6} \end{bmatrix}}_{\mathbf{E}^{(3)}} \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -6 \end{bmatrix}}_{\mathbf{E}^{(2)}\mathbf{E}^{(1)}\mathbf{A}} = \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(3)}\mathbf{E}^{(2)}\mathbf{E}^{(1)}\mathbf{A}}$$

Next, eliminate the nonzero elements in the third column on the second and first rows by multiplying, in sequence, by

$$\mathbf{E}^{(4)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{E}^{(5)} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

solution:

yielding the eliminated form

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(5)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(4)}} \underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(3)}\mathbf{E}^{(2)}\mathbf{E}^{(1)}\mathbf{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(5)}\mathbf{E}^{(4)}\mathbf{E}^{(3)}\mathbf{E}^{(2)}\mathbf{E}^{(1)}\mathbf{A}}$$

Finally, place a one on the diagonal in the second row by multiplying on the left by the elementary matrix which replaces the second row by $(1/2)R_2$, i.e:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(6)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(5)}\mathbf{E}^{(4)}\mathbf{E}^{(3)}\mathbf{E}^{(2)}\mathbf{E}^{(1)}\mathbf{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(6)}\mathbf{E}^{(5)}\mathbf{E}^{(4)}\mathbf{E}^{(3)}\mathbf{E}^{(2)}\mathbf{E}^{(1)}\mathbf{A}}$$

The original matrix has now been reduced to row-reduced echelon form.

b. Show that $\mathbf{A}^{-1}$ is precisely a product of the elementary matrices determined above.

solution:

Direct calculation MATLAB will verify that:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(6)}} \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(5)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(4)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{6} \end{bmatrix}}_{\mathbf{E}^{(3)}}$$

$$\cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}}_{\mathbf{E}^{(2)}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{\mathbf{E}^{(1)}} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{12} & -\frac{1}{6} \end{bmatrix}$$

solution:

and that this matrix is in fact $\mathbf{A}^{-1}$. This, of course, must be the case, since we've already showed, by direct calculation during the elimination process, that:

$$\mathbf{E}^{(6)}\mathbf{E}^{(5)}\mathbf{E}^{(4)}\mathbf{E}^{(3)}\mathbf{E}^{(2)}\mathbf{E}^{(1)}\ \mathbf{A} = \ \mathbf{I}$$

and from this equation, by definition:

$$\mathbf{E}^{(6)}\mathbf{E}^{(5)}\mathbf{E}^{(4)}\mathbf{E}^{(3)}\mathbf{E}^{(2)}\mathbf{E}^{(1)} = \mathbf{A}^{-1}$$

8. Solve the following system by $\mathbf{L}\,\mathbf{U}$ Decomposition (without row interchanges) and forward/backward substitution:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 8 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 1 \\ 3x_1 & - & 7x_2 & + & 4x_3 & = & 10 \end{array}$$

solution:

Original Matrix:

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} = \mathbf{U}_{work}$$

Eliminate in first column:

$$\begin{array}{lc} R_2 - (-1)R_1 & \longrightarrow \\ R_3 - (3)R_1 & \longrightarrow \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & -10 & -2 \end{bmatrix} \implies \mathbf{L}_{work} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & * & 1 \end{bmatrix}$$

Eliminate in second column:

$$R_3 - (10)R_2 \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & -52 \end{bmatrix} \implies \mathbf{L}_{work} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 10 & 1 \end{bmatrix}$$

Elimination/factorization completion.

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 10 & 1 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & -52 \end{bmatrix}$$

First perform forward substitution:

$$\mathbf{L}\,\mathbf{z} = \mathbf{b} \implies \begin{array}{rcrcrcr} z_1 & & & & & = & 8 \\ -z_1 & + & z_2 & & & = & 1 \\ 3z_1 & + & 10z_2 & + & z_3 & = & 10 \end{array}$$

or

$$z_1 = 8$$

$$z_2 = 1 + z_1 = 1 + (8) = 9$$

$$z_3 = 10 - 3z_1 - 10z_2 = 10 - 3(8) - (10)(9) = -104$$

solution:

(cont) Then back substitute

$$\mathbf{U}\,\mathbf{x} = \mathbf{z} \quad \Longrightarrow \quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 8 \\ & - & x_2 & + & 5x_3 & = & 9 \\ & & & - & 52x_3 & = & -104 \end{array}$$

so

$$x_3 = \frac{-104}{-52} = 2$$

$$x_2 = \frac{9 - 5x_3}{-1} = \frac{9 - (5)(2)}{-1} = \frac{-1}{-1} = 1$$

$$x_1 = \frac{8 - x_2 - 2x_3}{1} = \frac{8 - 1 - (2)(2)}{1} = \frac{3}{1} = 3$$

9. Determine which of the following are spanning sets for $\Re^3$. For those that are not, determine geometrically the dimension of the subspace of $\Re^3$ which they actually span.

a. $\{ [1 \quad 1 \quad 1] \}$

solution:

By definition, $\mathbf{b} \in span\left(\left\{\mathbf{a}^{(1)} , \mathbf{a}^{(2)} , \ldots , \mathbf{a}^{(r)}\right\}\right)$ if and only if

$$\alpha_1\mathbf{a}^{(1)} + \alpha_2\mathbf{a}^{(2)} + \ldots , \alpha_r\mathbf{a}^{(r)} = \mathbf{b}$$

is solvable. For vectors in $\Re^m$, this is equivalent to the condition that the system

$$\underbrace{\left[\mathbf{a}^{(1)} \vdots\, \mathbf{a}^{(2)} \vdots \ldots \vdots\, \mathbf{a}^{(r)}\right]}_{\mathbf{A}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{bmatrix} = \mathbf{b}$$

is consistent. For this example,

$$\left[\mathbf{A} \vdots \mathbf{b}\right] = \begin{bmatrix} 1 & \vdots & b_1 \\ 1 & \vdots & b_2 \\ 1 & \vdots & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \vdots & b_1 \\ 0 & \vdots & b_2 - b_1 \\ 0 & \vdots & b_3 - b_1 \end{bmatrix}$$

which is clearly solvable only if $b_1 = b_2 = b_3$. (This should be geometrically obvious!) So these do **not** span $\Re^3$. Also, by definition, in this case:

$$span\,(\{ [1 \quad 1 \quad 1] \}) = \{ [\alpha \quad \alpha \quad \alpha] \mid \alpha \text{ real}\}$$

which clearly defines a line through the origin. So the resulting subspace is one-dimensional.

b. { [1 0 0] , [0 0 1] }

solution:

These vectors will span $\Re^3$ if and only if

$$\left[\ \mathbf{A} \vdots \mathbf{b}\ \right] = \begin{bmatrix} 1 & 0 & \vdots & b_1 \\ 0 & 0 & \vdots & b_2 \\ 0 & 1 & \vdots & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \vdots & b_1 \\ 0 & 1 & \vdots & b_3 \\ 0 & 0 & \vdots & b_2 \end{bmatrix}$$

is always solvable. That is clearly not the case here, unless $b_2 = 0$. So these do **not** span $\Re^3$. But $b_2 = 0$ defines a vertical plane. Therefore the span here is a subspace of dimension two.

c. { [1 0 0] , [0 1 0] , [0 0 1] , [1 1 1] }

solution:

For this example, the vectors will span $\Re^3$ if and only if

$$\left[\ \mathbf{A} \vdots \mathbf{b}\ \right] = \begin{bmatrix} 1 & 0 & 0 & 1 & \vdots & b_1 \\ 0 & 1 & 0 & 1 & \vdots & b_2 \\ 0 & 0 & 1 & 1 & \vdots & b_3 \end{bmatrix}$$

is always solvable, which it clearly is (since the matrix is already in echelon form)! So these **do** span $\Re^3$.

d. $\{ [\ 1\ \ 2\ \ 1\ ] \,,\ [\ 2\ \ \ 0\ \ -1\ ] \,,\ [\ 4\ \ 4\ \ 1\ ] \}$

solution:

For this example, the vectors will span $\Re^3$ if and only if

$$\left[\ \mathbf{A} \vdots \mathbf{b}\ \right] = \left[\begin{array}{ccccc} 1 & 2 & 4 & \vdots & b_1 \\ 2 & 0 & 4 & \vdots & b_2 \\ 1 & -1 & 1 & \vdots & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc} 1 & 2 & 4 & \vdots & b_1 \\ 0 & -4 & -4 & \vdots & b_2 - 2b_1 \\ 0 & -3 & -3 & \vdots & b_3 - b_1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc} 1 & 2 & 4 & \vdots & b_1 \\ 0 & -4 & -4 & \vdots & b_2 - 2b_1 \\ 0 & 0 & 0 & \vdots & b_3 - \frac{3}{4}b_2 + \frac{1}{2}b_1 \end{array}\right]$$

is always solvable, which it clearly is not, unless $b_3 - \frac{3}{4}b_2 + \frac{1}{2}b_1 = 0$. So these do **not** span $\Re^3$. Moreover, the equation

$$b_3 - \frac{3}{4}b_2 + \frac{1}{2}b_1 = 0$$

clearly defines a plane in $\Re^3$. Therefore the span here is a two-dimensional subspace.

e. $\{ [\ 1\ \ 2\ \ 1\ ] \,,\ [\ 2\ \ \ 0\ \ -1\ ] \,,\ [\ 4\ \ 4\ \ 0\ ] \}$

solution:

For this example, the vectors will span $\Re^3$ if and only if

$$\left[\ \mathbf{A} \vdots \mathbf{b}\ \right] = \left[\begin{array}{ccccc} 1 & 2 & 4 & \vdots & b_1 \\ 2 & 0 & 4 & \vdots & b_2 \\ 1 & -1 & 0 & \vdots & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc} 1 & 2 & 4 & \vdots & b_1 \\ 0 & -4 & -4 & \vdots & b_2 - 2b_1 \\ 0 & 0 & -1 & \vdots & b_3 - \frac{3}{4}b_2 + \frac{1}{2}b_1 \end{array}\right]$$

is always solvable, which it clearly is. So these **do** span $\Re^3$.

10. a. Using the inner product, find the length of each of the following vectors:

(1.) $\mathbf{u} = [\,1 \quad 2 \quad 2\,]^T$

solution:

By definition, in $\Re^n$, $\|\,\mathbf{u}\,\| = \sqrt{\mathbf{u}^T\,\mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

Therfore, in this case

$$\|\,u\,\| = \sqrt{(1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3$$

(2.) $\mathbf{v} = [\ 1 \ \ 1 \ \ 3 \ -5]^T$

solution:

Proceeding as in part a.(1.):

$$\|\,v\,\| = \sqrt{(1)^2 + (1)^2 + (3)^2 + (-5)^2} = \sqrt{1+1+9+25} = \sqrt{36} = 6$$

(3.) $\mathbf{w} = \ [\ 1 \ \ 1 \ -2 \ \ \ 1 \ -1\ ]^T$

solution:

Again proceeding as in part a.(1.):

$$\begin{aligned}\|\,w\,\| &= \sqrt{(1)^2 + (1)^2 + (-2)^2 + (1)^2 + (-1)^2} \\ &= \sqrt{1+1+4+1+1} = \sqrt{8} = 2\sqrt{2}\end{aligned}$$

(Unfortunately, most vectors have lengths that are not exact squares!)

b. Find the angle between each of the following sets of vectors.

(1.) $\mathbf{u} = [\ 1\ \ 2\ \ 1\ ]^T$ and $\mathbf{v} = [\ 2\ \ \ 0\ \ -1\ ]^T$

solution:

By definition, in $\Re^n$, $\cos(\theta) = \dfrac{\mathbf{u}^T\,\mathbf{v}}{\|\,\mathbf{u}\,\|\|\,\mathbf{v}\,\|}$, where

$$\|\,u\,\| = \sqrt{\mathbf{u}^T\,\mathbf{v}} = \sqrt{u_1^2 + u_2^2 + \cdots u_n^2} \quad \text{and} \quad \mathbf{u}^{\mathrm{T}}\,\mathbf{v} = \mathrm{u}_1\mathrm{v}_1 + \mathrm{u}_2\mathrm{v}_2 \cdots + \mathrm{u}_n\mathrm{v}_n$$

So, in this case

$$\cos(\theta) = \frac{(1)(2) + (2)(0) + (1)(-1)}{\sqrt{(1)^2 + (2)^2 + (1)^2}\sqrt{(2)^2 + (0)^2 + (-1)^2}} = \frac{1}{\sqrt{30}}$$

and so, $\theta = \arccos\left(1/\sqrt{30}\right) \doteq 79.5^o$, i.e. they are almost perpendicular!

(2.) $\mathbf{u} = [\ 1\ \ 2\ \ 1\ \ 2\ \ 1\ ]^T$ and $\mathbf{w} = [\ 2\ \ 2\ \ 2\ \ 1\ \ 1\ ]^T$

solution:

Proceeding as in part b.(1.) above

$$\cos(\theta) = \frac{(1)(2) + (2)(2) + (1)(2) + (2)(1) + (1)(1)}{\sqrt{(1)^2 + (2)^2 + (1)^2 + (2)^2 + (1)^2}\sqrt{(2)^2 + (2)^2 + (2)^2 + (1)^2 + (1)^2}}$$
$$= \frac{11}{\sqrt{154}}$$

and so, $\theta = \arccos\left(11/\sqrt{154}\right) \doteq 27.6^o$, so they are somewhere between parallel and perpendicular.

11. Determine which of the following sets are orthogonal, orthonormal, or neither. Convert any that are orthogonal, but not orthonormal, to an equivalent orthonormal set.

a. $[\,1\ \ 1\ \ 1\,]^T$, $[\,2\ \ -1\ \ -1\,]$, $[\,1\ -2\ \ 1]^T$

solution:

By definition, a set of vectors $\mathbf{u}^{(i)}$ is orthogonal if and only if $\mathbf{u}^{(i)^T}\mathbf{u}^{(j)} = 0$ for all $i \neq j$. So here let

$$\mathbf{u}^{(1)} = [\,1\ \ 1\ \ 1\,]^T\ ,\quad \mathbf{u}^{(2)} = [\,2\ \ -1\ \ -1\,]\ ,\quad \text{and} \quad \mathbf{u}^{(3)} = [\,1\ -2\ \ 1]^T$$

Then

$$\mathbf{u}^{(1)^T}\mathbf{u}^{(2)} = (1)(2) + (1)(-1) + (1)(-1) = 0$$

and so $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are orthogonal. Similarly

$$\mathbf{u}^{(1)^T}\mathbf{u}^{(3)} = (1)(1) + (1)(-2) + (1)(1) = 0$$

and so $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(3)}$ are also orthogonal. But

$$\mathbf{u}^{(2)^T}\mathbf{u}^{(3)} = (2)(1) + (-1)(-2) + (-1)(1) = 3 \neq 0$$

and so $\mathbf{u}^{(2)}$ and $\mathbf{u}^{(3)}$ are **not** orthogonal. Since a **set** can be orthogonal if and only if every pair in the set is orthogonal, then this is **not** an orthogonal set.

b. $$\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} , \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} , \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} , \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$$

solution:

Proceeding as in part a., let

$$\mathbf{u}^{(1)} = \left[\ \frac{1}{2}\ \ \frac{1}{2}\ \ \frac{1}{2}\ \ \frac{1}{2}\ \right]^T$$

$$\mathbf{u}^{(2)} = \left[\ \frac{1}{2}\ \ -\frac{1}{2}\ \ \ \frac{1}{2}\ \ -\frac{1}{2}\ \right]^T\ , \qquad \text{etc.}$$

solution:

Then

$$\mathbf{u}^{(1)^T}\mathbf{u}^{(2)} = (1/2)(1/2) + (1/2)(-1/2) + (1/2)(1/2) + (1/2)(-1/2) = 0$$

Similar direct calculation shows that $\mathbf{u}^{(i)^T}\mathbf{u}^{(j)} = 0,\ \ i \neq j$, for all the other vectors in the set. So the set is orthogonal But, in addition

$$\begin{aligned}\|\,\mathbf{u}^{(1)}\,\| &= \sqrt{(1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2} \\ &= \sqrt{(1/4) + (1/4) + (1/4) + (1/4)} = \sqrt{1} = 1\end{aligned}$$

and similarly for the other $\mathbf{u}^{(i)}$. Therefore this set is not only orthogonal, but orthonormal. (In fact, it's an orthonormal basis for $\Re^4$.)

c. $\left\{ [\ 1\ \ 1\ \ 1\ \ 1\ ]^T\ ,\ [\ 1\ \ \ 1\ \ \ 3\ \ -5\ ]^T \right\}$

solution:

Proceeding as in part a., let

$$\mathbf{u}^{(1)} = [\ 1\ \ 1\ \ 1\ \ 1\ ]^T \quad \text{and} \quad \mathbf{u}^{(2)} = [\ 1\ \ \ 1\ \ \ 3\ \ -5\ ]^T$$

Then

$$\mathbf{u}^{(1)^T}\mathbf{u}^{(2)} = (1)(1) + (1)(1) + (1)(3) + (1)(-5) = 0$$

and so, since there are only two vectors in the set, it is an orthogonal set. However,

$$\|\,\mathbf{u}^{(1)}\,\| = \sqrt{(1)^2 + (1)^2 + (1)^2 + (1)^2} = \sqrt{4} = 2 \neq 1$$

and so the set is **not** orthonormal. However, since we can similarly show that $\|\,\mathbf{u}^{(2)}\,\| = 6$, then we can easily easily convert this to an equivalent orthonormal set by simple diving each vector by it's length.

!

solution:

Therefore, the set consisting of

$$\frac{\mathbf{u}^{(1)}}{\| \mathbf{u}^{(1)} \|} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \frac{\mathbf{u}^{(2)}}{\| \mathbf{u}^{(2)} \|} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{3}{6} \\ -\frac{5}{6} \end{bmatrix}$$

is an orthonormal one. (Note however that neither the original set, nor the normalize one can be a basis for $\Re^4$, but only for some two-dimensional subspace thereof.)

solution:

Then

$$\mathbf{u}^{(1)^T}\mathbf{u}^{(2)} = (1/2)(1/2) + (1/2)(-1/2) + (1/2)(1/2) + (1/2)(-1/2) = 0$$

Similar direct calculation shows that $\mathbf{u}^{(i)^T}\mathbf{u}^{(j)} = 0, \;\; i \neq j$, for all the other vectors in the set. So the set is orthogonal But, in addition

$$\begin{aligned}\| \mathbf{u}^{(1)} \| &= \sqrt{(1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2} \\ &= \sqrt{(1/4) + (1/4) + (1/4) + (1/4)} = \sqrt{1} = 1\end{aligned}$$

and similarly for the other $\mathbf{u}^{(i)}$. Therefore this set is not only orthogonal, but orthonormal. (In fact, it's an orthonormal basis for $\Re^4$.)

12. Find the dimension of and a basis for each of the fundamental subspaces associated with the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 3 & 1 \\ 2 & 1 & -2 & 5 & 3 \\ -2 & 1 & 6 & -6 & 1 \end{bmatrix}$$

solution:

For any right-hand side, the augmented matrix will have the form.

$$[\,\mathbf{A} \vdots \mathbf{b}\,] = \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 1 & b_1 \\ 2 & 1 & -2 & 5 & 3 & b_2 \\ -2 & 1 & 6 & -6 & 1 & b_3 \end{array}\right]$$

Elementary row operations will produce:

$$\begin{array}{l} \text{Pivot row} \rightarrow \\ \text{Pivot row} \rightarrow \\ \text{Pivot row} \rightarrow \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 1 & b_1 \\ 0 & 1 & 2 & -1 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 1 & 2 & b_3 - b_2 + 4b_1 \end{array}\right]$$

$$\begin{array}{cccc} \uparrow & \uparrow & & \uparrow \end{array}$$

Pivot Columns

Because the pivot rows of $\mathbf{U}$ can be used as a basis for Row($\mathbf{A}$) (Col($\mathbf{A}^H$)), then clearly the dimension of Row($\mathbf{A}$) is three and a basis is

$$\{[1 \quad 0 \quad -2 \quad 3 \quad 1]\ ,\ [0 \quad 1 \quad 2 \quad -1 \quad 1]\ ,\ [0 \quad 0 \quad 0 \quad 1 \quad 2]\}$$

Furthermore, because the columns of the original matrix corresponsing to the pivot columns of $\mathbf{U}$ can be used as a basis for Col($\mathbf{A}$), then clearly the dimension of the column space is also three, and a basis for it would consist of

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix} \right\}$$

solution:

The homogeneous problem is $\mathbf{A}\,\mathbf{x} = \mathbf{0}$. For this problem the echelon form of the augmented matrix is

$$\left[\begin{array}{rrrrrcr} 1 & 0 & -2 & 3 & 1 & \vdots & 0 \\ 0 & 1 & 2 & -1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 2 & \vdots & 0 \end{array}\right]$$

and since there are two free columns which can be set to arbitrary constants, back substitution will produce the general homogeneous solution form:

$$\begin{aligned} x_5 &= \alpha \quad \text{(arbitrary)} \\ x_4 &= -2\alpha \\ x_3 &= \beta \quad \text{(arbitrary)} \\ x_2 &= -3\alpha - 2\beta \\ x_1 &= 5\alpha + 2\beta \end{aligned} \qquad \Longrightarrow \qquad \mathbf{x} = \alpha \begin{bmatrix} 5 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and therefore the dimension of Null($\mathbf{A}$) is clearly two, and a basis for the subspace can be:

$$\left\{ \begin{bmatrix} 5 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} , \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

To determine Null($\mathbf{A}^H$), we first form the augmented matrix:

$$\left[\mathbf{A}^T \ \vdots \ \mathbf{0} \right] = \left[\begin{array}{rrrcr} 1 & 2 & -2 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ -2 & -2 & 6 & \vdots & 0 \\ 3 & 5 & -6 & \vdots & 0 \\ 1 & 3 & 1 & \vdots & 0 \end{array}\right]$$

solution:

Elementary row operations produce:

$$\left[\begin{array}{rrr:r} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ -2 & -2 & 6 & 0 \\ 3 & 5 & -6 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr:r} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr:r} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

and finally:

$$\left[\begin{array}{rrr:r} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, since there are obviously no free columns, Null($\mathbf{A}^H$) is of dimension zero, i.e.

$$\text{Null Space } \mathbf{A}^T = \{\mathbf{0}\}$$

To summarize the dimensional information:

Number of Rows in $\mathbf{A} \equiv m = 3$
Number of Columns in $\mathbf{A} \equiv n = 5$
Dimension of Row($\mathbf{A}$) =
Dimension of Col($\mathbf{A}^H$) = 3 (= rank)
Dimension of Col($\mathbf{A}$) = 3 (= rank)
Dimension of Null($\mathbf{A}$) = 2 (= n − rank)
Dimension of Null($\mathbf{A}^H$) = 0 (= m − rank)

(Note this is the same matrix as in problem 3.)

13. Find the dimension of and a basis for each of the fundamental subspaces associated with the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 4 & 4 \\ -3 & -6 & 3 & -5 & -8 \\ 2 & 4 & -2 & 5 & 2 \end{bmatrix}$$

solution:

For any right-hand side, the augmented matrix will have the form.

$$[\,\mathbf{A} \vdots \mathbf{b}\,] = \begin{bmatrix} 2 & 4 & -2 & 4 & 4 & \vdots & b_1 \\ -3 & -6 & 3 & -5 & -8 & \vdots & b_2 \\ 2 & 4 & -2 & 5 & 2 & \vdots & b_3 \end{bmatrix}$$

Elementary row operations will produce:

$$\begin{array}{r} \text{Pivot row} \rightarrow \\ \text{Pivot row} \rightarrow \\ \\ \end{array} \begin{bmatrix} 2 & 4 & -2 & 4 & 4 & \vdots & b_1 \\ 0 & 0 & 0 & 1 & -2 & \vdots & b_2 + \frac{3}{2} b_1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & b_3 - b_2 - \frac{5}{2} b_1 \end{bmatrix}$$
$$\begin{array}{ccccc} \uparrow & & & \uparrow & \\ \text{Pivot} & & \text{Columns} & & \end{array}$$

Elimination completion. Matrix is in echelon (upper triangular) form.

The pivot rows of $\mathbf{U}$ can be used as a basis for the row space, so:

$$\{\ [\ 2 \quad 4 \quad -2 \quad 4 \quad 4\ ], [\ 0 \quad 0 \quad 0 \quad 1 \quad -2\ ]\}$$

is a basis for the row space of $\mathbf{A}$, and the dimension is clearly two.

The columns of the original matrix corresponding to the pivot columns of $\mathbf{U}$ can be used as a basis for the column space, so:

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} , \begin{bmatrix} 4 \\ -5 \\ 5 \end{bmatrix} \right\}$$

is a basis for the column space of $\mathbf{A}$, and the dimension again is clearly two.

solution:

The homogeneous problem is $\mathbf{A}\,\mathbf{x} = \mathbf{0}$. For this problem the echelon form of the augmented matrix is

$$\left[\begin{array}{rrrrr:r} 2 & 4 & -2 & 4 & 4 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and we can backsolve to obtain: or equivalently:

$$\mathbf{x} = \begin{bmatrix} -6\alpha + \beta - 2\gamma \\ \gamma \\ \beta \\ 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -6 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where α, β and γ are arbitrary. But from this representation, we can immediately conclude that a basis for the null space is

$$\left\{ \begin{bmatrix} -6 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

and therefore the dimension of the null space is three.

To determine $\mathrm{Null}(\mathbf{A}^H)$, we first form the augmented matrix:

$$\left[\mathbf{A}^T \;\vdots\; \mathbf{0}\right] = \left[\begin{array}{rrr:r} 2 & -3 & 2 & 0 \\ 4 & -6 & 4 & 0 \\ -2 & 3 & -2 & 0 \\ 4 & -5 & 5 & 0 \\ 4 & -8 & 2 & 0 \end{array}\right]$$

solution:

Elementary row operations produce:

$$\left[\begin{array}{rrrcr} 2 & -3 & 2 & \vdots & 0 \\ 4 & -6 & 4 & \vdots & 0 \\ -2 & 3 & -2 & \vdots & 0 \\ 4 & -5 & 5 & \vdots & 0 \\ 4 & -8 & 2 & \vdots & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrcr} 2 & -3 & 2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & -2 & -2 & \vdots & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrcr} 2 & -3 & 2 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$$

and therefore, by back substitution, the homogeneous solution is:

$$\begin{aligned} x_3 &= \alpha \\ x_2 &= -\alpha \\ x_1 &= -\frac{5}{2}\alpha \end{aligned} \quad \Longrightarrow \quad \mathbf{x} = \alpha \begin{bmatrix} -\frac{5}{2} \\ -1 \\ 1 \end{bmatrix}$$

and therefore, obviously, the null space of $\mathbf{A}^T$ is of dimension one, with basis:

$$\left\{ \begin{bmatrix} -\frac{5}{2} \\ -1 \\ 1 \end{bmatrix} \right\}$$

To summarize the dimensional information:

Number of Rows in $\mathbf{A} \equiv m = 3$

Number of Columns in $\mathbf{A} \equiv n = 5$

Dimension of Row($\mathbf{A}$) =

Dimension of Col($\mathbf{A}^H$) = 2 (= rank)

Dimension of Col($\mathbf{A}$) = 2 (= rank)

Dimension of Null($\mathbf{A}$) = 3 (= n − rank)

Dimension of Null($\mathbf{A}^H$) = 1 (= m − rank)

(Note this is the same matrix as in problem 4.)

14. Find the coordinates of

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

in terms of the ordered basis

$$\mathbf{B} = \left\{ \mathbf{b}^{(1)} \ , \ \mathbf{b}^{(2)} \right\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$$

and confirm your answer geometrically.

solution:

By definition, in $\Re^m$, $[\,\mathbf{x}\,]_{\mathbf{B}}$, the coordinates of $\mathbf{x}$ with respect to the ordered basis $\mathbf{B}$ satisfies the relationship

$$\mathbf{x} = \mathbf{B}\,[\,\mathbf{x}\,]_{\mathbf{B}}$$

where $\mathbf{x}$ on the left here is expressed in terms of the standard coordinates, and the columns of the matrix $\mathbf{B}$ are precisely the basis vectors (also expressed in terms of standard coordinates). Therefore for this problem, we must simply solve

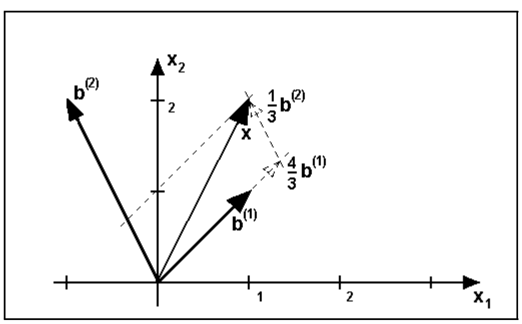
$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} [\mathbf{x}]_{\mathbf{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Longrightarrow \quad [\mathbf{x}]_{\mathbf{B}} = \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}$$

Geometrically, we interpret this to mean that to reach the "tip" of $\mathbf{x}$, we must

(i) First move (4/3) the length of $\mathbf{b}^{(1)}$ in the direction of $\mathbf{b}^{(1)}$,

(ii) Then move (1/3) the length of $\mathbf{b}^{(2)}$ in the direction of $\mathbf{b}^{(2)}$,

The below figure confirms that to be exactly the case here:



15. Consider the transformation specified by

$$\mathbf{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_3 + x_4 \\ 2x_1 - 3x_2 + 3x_4 \\ x_3 - x_1 \end{bmatrix}$$

Find the coordinate matrix of $\mathbf{T}$ relative to the standard bases for its "input" and "output" spaces.

solution:

The standard bases for $\Re^4$ (the "input" space) is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The "outputs" for each of these "input" basis vectors are, respectively:

$$\mathbf{T}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{T}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \quad \mathbf{T}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

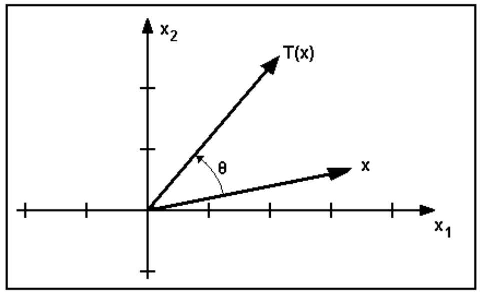
and

$$\mathbf{T}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

and therefore

$$\mathbf{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 3 & 0 & -1 & 1 \\ 2 & -3 & 0 & 3 \\ -1 & 0 & 1 & 0 \end{bmatrix}}_{[\ \mathbf{T}\ ]} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

16. Consider the linear transformation $\mathbf{T}(\ )$ from $\Re^2$ to $\Re^2$ that simply rotates each "input" vector through a fixed, specified angle (θ), i.e.



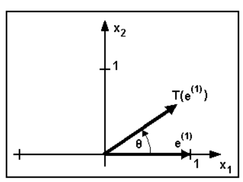
Find the coordinate matrix for this transformation relative to the standard bases for its "input" and "output" spaces.

solution:

The standard basis for $\Re^2$ (the "input" space) is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

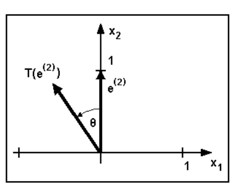
For the first standard basis vector, $\mathbf{e}^{(1)} = [\ 1\ \ 0\ ]^T$, the "output" can, from the following figure,



clealy seen to be

$$\mathbf{T}\left(\mathbf{e}^{(1)}\right) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Similarly, for $\mathbf{e}^{(2)} = [\ 0\ \ 1\ ]^T$, the figure



shows that

$$\mathbf{T}\left(\mathbf{e}^{(2)}\right) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

solution:

Therefore, the coordinate matrix for $\mathbf{T}$ relative to the standard bases for its "input" and "output" spaces. is

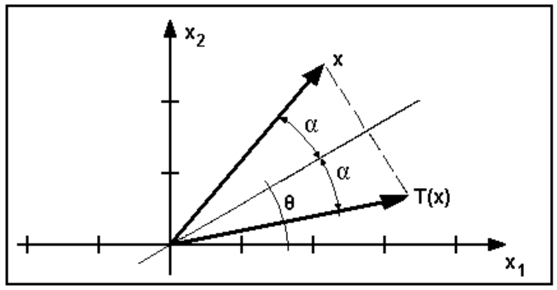
$$[\mathbf{T}] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

or, as it is often shown

$$[\mathbf{T}] = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

where $c = \cos(\theta)$, and $s = \sin(\theta)$.

17. Consider the transformation that reflects any given "input" vector about a specified line at an angle of θ with respect to the x_1 axis, i.e.:



Show that the coordinate matrix for this transformation relative to the standard bases for its "input" and "output" spaces is:

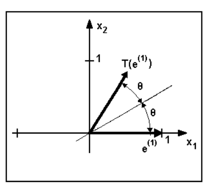
$$[\mathbf{T}] = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

solution:

The standard basis for $\Re^2$ (the "input" space) is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

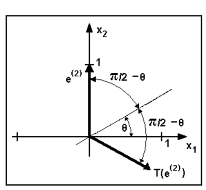
For the first standard basis vector, $\mathbf{e}^{(1)} = [\ 1 \ \ 0\ ]^T$, the "output" can, from the following figure,



clealy seen to be

$$\mathbf{T}\left(\mathbf{e}^{(1)}\right) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$$

Similarly, for $\mathbf{e}^{(2)} = [\ 0 \ \ 1\ ]^T$, the figure



solution:

along with a little basic trigonometry, shows that

$$\mathbf{T}\left(\mathbf{e}^{(2)}\right) = \begin{bmatrix} \cos(2\theta - \pi/2) \\ \sin(2\theta - \pi/2) \end{bmatrix} = \begin{bmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{bmatrix}$$

Therefore, the coordinate matrix for $\mathbf{T}$ relative to the standard bases for its "input" and "output" spaces. is

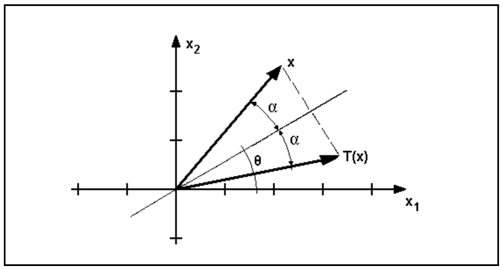
$$[\mathbf{T}] = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

or, as it is often shown

$$[\mathbf{T}] = \begin{bmatrix} (c^2 - s^2) & 2cs \\ 2cs & (s^2 - c^2) \end{bmatrix}$$

where $c = \cos(\theta)$, and $s = \sin(\theta)$.

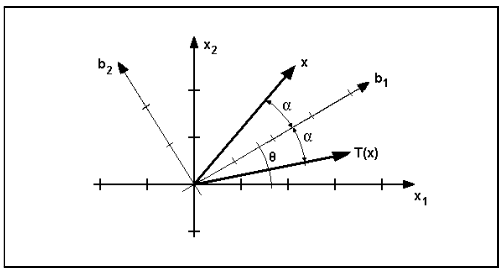
18. Consider the transformation that reflects any given "input" vector about a specified line at an angle of θ with respect to the x_1 axis, i.e.:



We have previously shown that, in terms of the standard (i.e. x_1, x_2) coordinate system, the matrix for this transformation is:

$$[\mathbf{T}] = \begin{bmatrix} (c^2 - s^2) & 2cs \\ 2cs & (s^2 - c^2) \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$. Consider now the non-standard coordinate system oriented at an angle of θ relative to the standard one, i.e.



Using the standard formula for similarity transformations, i.e.

$$[\mathbf{T}]_{\mathbf{B}} = \mathbf{B}^{-1}\,[\mathbf{T}]\,\mathbf{B}$$

find the matrix for this transformation relative to the above non-standard basis, and interpret your result geometrically.

solution:

In the standard formula for similarity transformations, the columns of the matrix $\mathbf{B}$ are the basis for the non-standard basis, *expressed in terms of standard coordinates.* For this problem, the unit vectors corresponding to the b_1 and b_2 axes are geometrically easily seen to be:

$$\mathbf{b}^{(1)} = \begin{bmatrix} c \\ s \end{bmatrix} \quad \text{and} \quad \mathbf{b}^{(2)} = \begin{bmatrix} -s \\ c \end{bmatrix}$$

solution:

Therefore

$$\mathbf{B} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

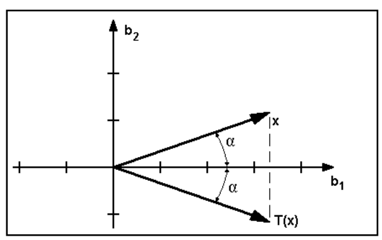
and therefore,

$$\begin{aligned}
[\mathbf{T}]_{\mathbf{B}} = \mathbf{B}^{-1}\,[\mathbf{T}]\,\mathbf{B} &= \begin{bmatrix} c & -s \\ s & c \end{bmatrix}^{-1} \begin{bmatrix} (c^2-s^2) & 2cs \\ 2cs & (s^2-c^2) \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \\
&= \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} (c^2-s^2) & 2cs \\ 2cs & (s^2-c^2) \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \\
&= \begin{bmatrix} (c^2+s^2) & 0 \\ 0 & (-s^2-c^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}$$

where we have used the trigonometric identity

$$c^2 + s^2 \equiv \cos^2(\theta) + \sin^2(\theta) = 1$$

But now observe that, relative to an observer who believe that the b_1 axis is horizontal, the transformation appear to behave as



i.e., the horizontal coordinate of the "output" "input" is unchanged from that of the "input," and the vertical component of the "output" has the same magnitude, but the opposite sign of the corresponding component of the "input." But this means that, from the viewpoint of this observer, the matrix for the transformation should be precisely:

$$[\mathbf{T}]_{\mathbf{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which is exactly what we computed using the rule: $[\mathbf{T}]_{\mathbf{B}} = \mathbf{B}^{-1}\,[\mathbf{T}]\,\mathbf{B}$.

19. For each of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 1 & 4 \\ 0 & -1 & 5 \end{bmatrix} \quad , \quad \mathbf{E} = \begin{bmatrix} 0 & 6 & 3 \\ -1 & 5 & 1 \\ -1 & 2 & 4 \end{bmatrix} \quad , \quad \mathbf{F} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(i) Find all of the eigenvalues and their associated multiplicities.

(ii) For each different eigenvalue, find the dimension of and a basis for the associated eigenspace.

(iii) Determine whether the matrix is diagonalizable.

solution:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix} \implies \mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{bmatrix}$$

$$\begin{aligned} \implies \quad P_3(\lambda) &= \det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \\ &= -(\lambda-1)(\lambda-2)(\lambda-3) \quad \implies \quad \lambda = 1, 2, 3 \end{aligned}$$

Now, for $\lambda_1 = 1$

$$(\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{v}^{(1)} = 0 \longrightarrow \left[\begin{array}{ccc:c} 2 & 2 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ -2 & -4 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 2 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix clearly has only one free column Therefore there will be only one linearly independent eigenvector, and so the associated eigenspace will be of dimension one. Direct computation produces:

$$\mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

which is, by definition, a basis for the eigenspace associated with $\lambda_1 = 1$.

solution:

Similarly, for $\lambda_2 = 2$

$$(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{v}^{(2)} = 0 \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ -2 & -4 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Longrightarrow \quad \mathbf{v}^{(2)} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Again (and predictably, since λ_2 is distinct), the associated eigenspace is of dimension one, and the eigenvector is a basis.

Lastly, for $\lambda_3 = 3$

$$(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{v}^{(3)} = 0 \longrightarrow \left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & -4 & -4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Longrightarrow \quad \mathbf{v}^{(3)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

and so the eigenspace associated with λ_3 is also of dimension one. The matrix of eigenvectors is

$$\mathbf{V} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Therefore, since these eigenvectors are linearly independent (predictable as soon as the eigenvalues are known to be distinct), then:

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(This can be verified by MATLAB!)

solution:

For part b:

$$\mathbf{B} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{B} - \lambda\mathbf{I} = \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$$

$$\begin{aligned} \Longrightarrow \quad P_3(\lambda) = \det(\mathbf{B} - \lambda\mathbf{I}) &= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 \\ &= -(\lambda-8)(\lambda+1)^2 \quad \Longrightarrow \quad \lambda = 8, -1, -1 \end{aligned}$$

Now, for $\lambda_1 = 8$ (which is distinct and therefore will have an associated eigenspace of dimension one):

$$(\mathbf{B} - \lambda_1\mathbf{I})\,\mathbf{v}^{(1)} = 0 \;\longrightarrow\; \left[\begin{array}{rrr|r} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} -5 & 2 & 4 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Longrightarrow \quad \mathbf{v}^{(1)} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

But, for $\lambda_2 = \lambda_3 = -1$, which is of multiplicity two

$$(\mathbf{B} - \lambda_2\mathbf{I})\,\mathbf{v}^{(2)} = 0 \;\longrightarrow\; \left[\begin{array}{rrr|r} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this instance, we have two free columns, and therefore two linearly independent eigenvectors and an eigenspace of dimension two:

$$\mathbf{v}^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}^{(3)} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{V} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

solution:

Since we have three linearly independent eigenvectors, even though there was a multiple eigenvalue, then (again as can be verified by MATLAB):

$$\mathbf{V}^{-1}\mathbf{B}\mathbf{V} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

For part c:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{C} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{bmatrix}$$

$$\Longrightarrow \quad P_3(\lambda) = \det(\mathbf{C} - \lambda\mathbf{I}) = (1-\lambda)(\lambda^2 - 2\lambda + 5)$$

$$\Longrightarrow \quad \lambda = 1, 1 \pm 2i$$

Now, for $\lambda_1 = 1$

$$(\mathbf{C} - \lambda_1\mathbf{I})\,\mathbf{v}^{(1)} = 0 \;\longrightarrow\; \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Longrightarrow \quad \mathbf{v}^{(1)} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

and, for $\lambda_2 = 1 + 2i$

$$(\mathbf{C} - \lambda_2\mathbf{I})\mathbf{v}^{(2)} = 0 \;\longrightarrow\; \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 2 & -2i & -2 & 0 \\ 3 & 2 & -2i & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} -2i & 0 & 0 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Longrightarrow \quad \mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

solution:

However, since $\mathbf{C}$ is real, eigenvalues and eigenvectors appear in conjugate pairs, i.e.

$$\lambda_3 = 1 - 2i \quad \Longrightarrow \quad \mathbf{v}^{(3)} = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Since the eigenvalues are distinct (the fact they're complex is irrelevant), then all of the eigenspaces are of dimension one and the eigenvectors are linearly independent. Therfore

$$\mathbf{V} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & i & -i \\ 2 & 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{V}^{-1}\mathbf{C}\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{bmatrix}$$

For part d:

$$\mathbf{D} = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 1 & 4 \\ 0 & -1 & 5 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{D} - \lambda\mathbf{I} = \begin{bmatrix} 3-\lambda & -2 & 5 \\ 0 & 1-\lambda & 4 \\ 0 & -1 & 5-\lambda \end{bmatrix}$$

$$\Longrightarrow \quad P_3(\lambda) = \det(\mathbf{D} - \lambda\mathbf{I}) = (3-\lambda)(\lambda^2 - 6\lambda + 9)$$

$$= -(\lambda - 3)^3 \quad \Longrightarrow \quad \lambda = 3, 3, 3$$

Now, for the triple root $\lambda_1 = \lambda_2 = \lambda_3 = 3$

$$(\mathbf{D} - \lambda_1\mathbf{I})\,\mathbf{v}^{(1)} = 0 \longrightarrow \left[\begin{array}{ccc:c} 0 & -2 & 5 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 0 & -2 & 5 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here only one free column. Therefore there the associated eigenspace will be only one dimensional, and only one linearly independent eigenvector exists:

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Consequently $\mathbf{D}$ will **not** be diagonalizable.

solution:

For part e:

$$\mathbf{E} = \begin{bmatrix} 0 & 6 & 3 \\ -1 & 5 & 1 \\ -1 & 2 & 4 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{E} - \lambda \mathbf{I} = \begin{bmatrix} -\lambda & 6 & 3 \\ -1 & 5-\lambda & 1 \\ -1 & 2 & 4-\lambda \end{bmatrix}$$

$$\Longrightarrow \quad P_3(\lambda) = \det(\mathbf{E} - \lambda \mathbf{I}) = -\lambda^3 + 9\lambda^2 - 27\lambda + 27$$

$$= -(\lambda - 3)^3 \quad \Longrightarrow \quad \lambda = 3, 3, 3$$

Now, for $\lambda_1 = \lambda_2 = \lambda_3 = 3$

$$(\mathbf{E} - \lambda_1 \mathbf{I})\, \mathbf{v}^{(1)} = 0 \longrightarrow \left[\begin{array}{ccc|c} -3 & 6 & 3 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here we have two free columns. Therefore the associated eigenspace will be only two-dimensional, and only two linearly independent eigenvectors exist:

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Consequently **E** will **not** be diagonalizable.

Finally, for part f:

$$\mathbf{F} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{F} - \lambda \mathbf{I} = \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$\Longrightarrow \quad P_3(\lambda) = \det(\mathbf{F} - \lambda \mathbf{I}) = (3 - \lambda)^3 \quad \Longrightarrow \quad \lambda = 3, 3, 3$$

Now, for the triple root $\lambda_1 = \lambda_2 = \lambda_3 = 3$

$$(\mathbf{F} - \lambda_1 \mathbf{I})\, \mathbf{v}^{(1)} = 0 \longrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

solution:

Here there are no pivots, and three free columns. Hence the eigenspace associated with λ_1 will be of the same dimension as the multiplicity of the root. So there are three linearly independent eigenvectors:

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \text{and} \quad \mathbf{v}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and therefore

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \mathbf{I} \quad \Longrightarrow \quad \mathbf{V}^{-1}\mathbf{F}\mathbf{V} \equiv \mathbf{F}$$

and $\mathbf{F}$ is trivially diagonalizable (because it's already diagonal).

20. For each of the following symmetric matrices:

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 2 & -2 & 2 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & 5 \\ -5 & 5 & 7 \end{bmatrix}$$

(i) Find all of the eigenvalues and their associated multiplicities.

(ii) For each different eigenvalue, find the dimension of and an orthonormal basis for the associated eigenspace.

(iii) Determine whether the matrix is diagonalizable.

solution:

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 2 \\ 4 & -1 & -2 \\ 2 & -2 & 2 \end{bmatrix} \implies \mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -1-\lambda & 4 & 2 \\ 4 & -1-\lambda & -2 \\ 2 & -2 & 2-\lambda \end{bmatrix}$$

$$\begin{aligned} \implies \quad P_3(\lambda) &= \det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda^3 + 27\lambda - 54 \\ &= -(\lambda+6)(\lambda-3)^2 \quad \implies \quad \lambda = -6, 3, 3 \end{aligned}$$

Now, for $\lambda_1 = -6$

$$(\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{v}^{(1)} = 0 \longrightarrow \left[\begin{array}{ccc|c} 5 & 4 & 2 & 0 \\ 4 & 5 & -2 & 0 \\ 2 & -2 & 8 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 5 & 4 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

As expected (distinct eigenvalue), this matrix has only one free column and therefore only one linearly independent eigenvector and an associated eigenspace of dimension one. Direct computation produces:

$$\hat{\mathbf{v}}^{(1)} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \quad \text{or, when normalized} \quad \mathbf{v}^{(1)} = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

which is, by definition, a basis for the eigenspace associated with $\lambda_1 = -6$.

solution:

Similarly, for the double root $\lambda_2 = \lambda_3 = 3$

$$(\mathbf{A} - \lambda_2 \mathbf{I})\,\mathbf{v}^{(2)} = 0 \longrightarrow \left[\begin{array}{ccc:c} -4 & 4 & 2 & 0 \\ 4 & -4 & -2 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} -4 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have two free columns here, and therefore (predictably, since **A** is symmetric), the eigenspace associated with the double root is of dimension two. Normal Gaussian elimination methods would produce the two eigenvectors

$$\tilde{\mathbf{v}}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{v}}^{(3)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

as a basis. Unfortunately, these are not orthogonal. However, using Gram-Schmidt, we can easily find an equivalent, orthogonal set:

$$\hat{\mathbf{v}}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{v}}^{(3)} = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$$

or, when normalized

$$\mathbf{v}^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad \mathbf{v}^{(3)} = \begin{bmatrix} \frac{4}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \\ -\frac{2}{\sqrt{45}} \end{bmatrix}$$

(Note there are other choices as well, some of which look "prettier.") Since **A** has a full set of linearly independent eigenvectors, then **A** is diagonalizable. In fact, in this case, since

$$\mathbf{V} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ \frac{2}{3} & 0 & \frac{5}{\sqrt{45}} \\ \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \end{bmatrix}$$

solution:

is an orthogonal matrix (i.e. $\mathbf{V}^T\mathbf{V} = \mathbf{I}$), then

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} \;\equiv\; \mathbf{V}^T\mathbf{A}\mathbf{V} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

For part b:

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & -5 \\ 0 & 2 & 5 \\ -5 & 5 & 7 \end{bmatrix} \quad\Longrightarrow\quad \mathbf{B} - \lambda\mathbf{I} = \begin{bmatrix} 2-\lambda & 0 & -5 \\ 0 & 2-\lambda & 5 \\ -5 & 5 & 7-\lambda \end{bmatrix}$$

$$\begin{aligned} \Longrightarrow \quad P_3(\lambda) &= \det(\mathbf{B} - \lambda\mathbf{I}) = -\lambda^3 + 11\lambda^2 + 18\lambda - 72 \\ &= -(\lambda - 12)(\lambda - 2)(\lambda + 3) \quad \Longrightarrow \quad \lambda = -3, 2, 12 \end{aligned}$$

Now, for $\lambda_1 = -3$

$$(\mathbf{B} - \lambda_1\mathbf{I})\,\mathbf{v}^{(1)} = 0 \;\longrightarrow\; \left[\begin{array}{ccc|c} 5 & 0 & -5 & 0 \\ 0 & 5 & 5 & 0 \\ -5 & 5 & 10 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 5 & 0 & -5 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

As expected (distinct eigenvalue), this matrix has only one free column and therefore only one linearly independent eigenvector and an associated eigenspace of dimension one. Direct computation produces:

$$\hat{\mathbf{v}}^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{or, when normalized} \quad \mathbf{v}^{(1)} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Similarly, for $\lambda_2 = 2$

$$(\mathbf{B} - \lambda_2\mathbf{I})\,\mathbf{v}^{(2)} = 0 \;\longrightarrow\; \left[\begin{array}{ccc|c} 0 & 0 & -5 & 0 \\ 0 & 0 & 5 & 0 \\ -5 & 5 & 10 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -5 & 5 & 10 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

solution:

Again, as expected, this matrix has only one free column and therefore only one linearly independent eigenvector and an associated eigenspace of dimension one. Direct computation produces:

$$\hat{\mathbf{v}}^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{or, when normalized} \quad \mathbf{v}^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Lastly, for $\lambda_3 = 12$

$$(\mathbf{B} - \lambda_3 \mathbf{I})\mathbf{v}^{(3)} = 0 \longrightarrow \left[\begin{array}{rrr:r} -10 & 0 & -5 & 0 \\ 0 & -10 & 5 & 0 \\ -5 & 5 & -5 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrr:r} -10 & 0 & -5 & 0 \\ 0 & -10 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

As again expected, this matrix has only one free column and therefore only one linearly independent eigenvector and an associated eigenspace of dimension one. Direct computation produces:

$$\hat{\mathbf{v}}^{(3)} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \quad \text{or, when normalized} \quad \mathbf{v}^{(3)} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Finally, and again as expected, we have a complete set of orthonormal eigenvectors, and if we use the orthogonal matrix

$$\mathbf{V} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}, \quad \text{then} \quad \mathbf{V}^{-1}\mathbf{B}\mathbf{V} \equiv \mathbf{V}^T\mathbf{B}\mathbf{V} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

i.e. $\mathbf{V}$ will diagonalize $\mathbf{B}$.

21. Consider the matrices:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 5 \\ 0 & -1 & 3 \\ -2 & 3 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(i) Find $\mathbf{S}^{-1}$.

solution:

First form the augmented matrix:

$$\left[\mathbf{S} \ \vdots \ \mathbf{I}\right] = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and proceed with Gauss-Jordan elimination:

$$\begin{array}{c} \\ R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{1}{2}R_1 \end{array} \longrightarrow \begin{bmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \\ \\ R_3 + R_2 \end{array} \longrightarrow \begin{bmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

Back substitute:

$$\begin{array}{c} R_1 - R_3 \\ R_2 - \frac{1}{2}R_3 \\ \\ \end{array} \longrightarrow \begin{bmatrix} 2 & 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} \frac{1}{2}R_1 \\ -R_2 \\ \\ \end{array} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

and so

$$\mathbf{S}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix}$$

(ii) Find $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$

solution:

$$\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 5 \\ 0 & -1 & 3 \\ -2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 5 \\ 2 & 4 & 2 \\ 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii) Show that $\mathbf{A}$ and $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$ have the same characteristic polynomial (and hence the same eigenvalues).

solution:

The characteristic polynomial for $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{vmatrix} -1-\lambda & 1 & 5 \\ 0 & -1-\lambda & 3 \\ -2 & 3 & 3-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\det \begin{vmatrix} -1-\lambda & 3 \\ 3 & 3-\lambda \end{vmatrix} + (-2)\det \begin{vmatrix} 1 & 5 \\ -1-\lambda & 3 \end{vmatrix}$$

$$= (-1-\lambda)(\lambda^2 - 2\lambda - 12) - 2\,(5\lambda + 8) = -\lambda^3 + \lambda^2 + 4\lambda - 4$$

while the characteristic polynomial for $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$ is

$$\det\left(\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S} - \lambda\mathbf{I}\right) = \det \begin{vmatrix} 2-\lambda & 2 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-2-\lambda)(1-\lambda) = -\lambda^3 + \lambda^2 + 4\lambda - 4$$

(iv) Find the eigenvalues and eigenvectors of $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$.

solution:

For $\quad \mathbf{S}^{-1}\mathbf{A}\,\mathbf{S} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ the eigenvalues are clearly $\lambda = -2, 2, 1$

For $\lambda_1 = -2$,

$$\left(\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S} - \lambda_1 \mathbf{I}\right)\mathbf{v}^{(1)} = \mathbf{0} \quad \Longrightarrow \quad \begin{bmatrix} 4 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 2$,

$$\left(\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S} - \lambda_2 \mathbf{I}\right)\mathbf{v}^{(2)} = \mathbf{0} \quad \Longrightarrow \quad \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

And for $\lambda_3 = 1$,

$$\left(\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S} - \lambda_3 \mathbf{I}\right)\mathbf{v}^{(3)} = \mathbf{0} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{v}^{(3)} = \begin{bmatrix} -8 \\ 1 \\ 3 \end{bmatrix}$$

(v) Show by direct computation that, if $\mathbf{v}^{(i)}$ is an eigenvector of $\mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$ corrensponding to λ_i, then $\mathbf{w}^{(i)} = \mathbf{S}\mathbf{v}^{(i)}$ is an eigenvector of $\mathbf{A}$ corresponding to the same eigenvalue.

solution:

For $\mathbf{v}^{(1)}$: $\quad \mathbf{w}^{(1)} = \mathbf{S}\mathbf{v}^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

and so

$$\mathbf{A}\mathbf{w}^{(1)} = \begin{bmatrix} -1 & 1 & 5 \\ 0 & -1 & 3 \\ -2 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = -2\,\mathbf{w}^{(1)}$$

solution:

Therefore, by defintion, $\mathbf{w}^{(1)}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_1 = -2$.

Similarly, for $\mathbf{v}^{(2)}$: $$\mathbf{w}^{(2)} = \mathbf{S}\mathbf{v}^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

and so $$\mathbf{A}\mathbf{w}^{(2)} = \begin{bmatrix} -1 & 1 & 5 \\ 0 & -1 & 3 \\ -2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 2\,\mathbf{w}^{(2)}$$

Therefore, by defintion, $\mathbf{w}^{(2)}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_2 = 2$.

Finally, for $\mathbf{v}^{(3)}$: $$\mathbf{w}^{(3)} = \mathbf{S}\mathbf{v}^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -13 \\ -6 \\ -4 \end{bmatrix}$$

and so $$\mathbf{A}\mathbf{w}^{(2)} = \begin{bmatrix} -1 & 1 & 5 \\ 0 & -1 & 3 \\ -2 & 3 & 3 \end{bmatrix} \begin{bmatrix} -13 \\ -6 \\ -4 \end{bmatrix} = \begin{bmatrix} -13 \\ -6 \\ -4 \end{bmatrix} = \mathbf{w}^{(3)}$$

Therefore, by defintion, $\mathbf{w}^{(3)}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda_3 = 1$.